Membranes at quantum criticality

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Abstract: We propose a quantum theory of membranes designed such that the ground-state wavefunction of the membrane with compact spatial topology Σ_h reproduces the partition function of the bosonic string on worldsheet Σ_h. The construction involves worldvolume matter at quantum criticality, described in the simplest case by Lifshitz scalars with dynamical critical exponent $z = 2$. This matter system must be coupled to a novel theory of worldvolume gravity, also exhibiting quantum criticality with $z = 2$. We first construct such a nonrelativistic "gravity at a Lifshitz point" with $z = 2$ in $D+1$ spacetime dimensions, and then specialize to the critical case of $D = 2$ suitable for the membrane worldvolume. We also show that in the second-quantized framework, the string partition function is reproduced if the spacetime ground state takes the form of a Bose-Einstein condensate of membranes in their first-quantized ground states, correlated across all genera.

Contents

1 Introduction

In the democracy of all branes, strings seem to occupy a privileged position, for a variety of reasons. One argument suggesting that strings are unique among all branes points out the

apparent nonexistence of the analog of string perturbation theory for membranes. As discussed in [1], this argument itself is related to several distinct phenomena. First, quantum gravity is at its critical dimension on the two-dimensional string worldsheet, leading to a sensible worldsheet quantum theory at each fixed order in the string perturbation expansion. In contrast, no clear quantization technique is available to make sense of quantum gravity coupled to matter on higher-dimensional worldvolumes, at least within the conventional approach of renormalizable quantum field theory. Secondly, while two-dimensional worldsheets can be organized in terms of a simple discrete invariant — the genus — which counts the loops of diagrams, no such simple classification is available for membranes. This fact is traditionally interpreted as an indication that if a quantum theory of membranes existed, it would have to be strongly coupled. This in turn implies that even if the worldvolume theory on a fixed topology were well-defined, we would not know how to sum over distinct topologies.

In this paper, we explore the possibility of constructing a new worldvolume quantum theory of gravity and matter in $2 + 1$ dimensions, at least in the simplest case of a bosonic theory. The price we pay for the avoidance of some of the above-mentioned obstacles is a strong anisotropy between space and time in the worldvolume theory, a phenomenon familiar from the study of condensed matter systems at quantum criticality, dynamical critical phenomena, and in statistical dynamics of systems far from equilibrium. In the process, we will uncover a new class of gravity theories with anisotropic scaling between space and time, characterized by a nontrivial dynamical critical exponent z. Such nonrelativistic gravity models can clearly be of broader interest beyond $2 + 1$ dimensional worldvolumes, and we introduce them first in section 4 in the general case of $D + 1$ spacetime dimensions, before specializing to $D = 2$.

We begin by posing an auxiliary problem: *Can we find a quantum theory of membranes, such that its ground-state wavefunction reproduces the partition function of the bosonic string?* This type of question — about the existence of two systems in such a relationship to each other — is central to many areas of physics, primarily with applications to condensed matter. For example, one might start with a universality class describing an equilibrium system in D dimensions at criticality, and ask how the critical behavior extends to the dynamical phenomena in $D + 1$ dimensions. Requiring that in the static limit one recovers the partition function of the original D-dimensional equilibrium system is effectively equivalent to the type of question that we ask above. Essentially the same logic has been used in recent years to produce new interesting classes of quantum critical systems in $D + 1$ dimensions, starting from known classical universality classes in D dimensions. In stochastic quantization, one asks a similar question in imaginary time: The task is to build a nonequilbrium system in $D + 1$ dimensions which relaxes at late times to its ground state, which reproduces the partition function of a D-dimensional system one is interested in. The techniques that we use in our construction of gravity models are closely related to the methods used in these areas of condensed matter theory. Similar ideas have been applied to Yang-Mills gauge theories in [2].

Additional motivation for asking our auxiliary question comes also in part from the recent findings in topological string theory [3, 4], the OSV conjecture [5], topological M-

theory [6], and noncritical M-theory [7, 8]. In that context, interesting relationships have been discovered in which the partition function of one theory is related to a wavefunction of another theory in a higher dimension, and one naturally wonders whether such connections are more prevalent in the general context of string and M-theory.

2   The second-quantized theory

In this section, we first define the auxiliary problem a little more precisely. Then, we will assume that the problem is solved at the level of first quantization, i.e., that we can construct a membrane worldvolume theory whose ground-state wavefunction for the membrane of spatial topology Σ_h reproduces the partition function of the bosonic string on Σ_h. Given this assumption, we will show how to solve the problem at the second-quantized level, in the Hilbert space of multi-membrane states. An attempt to solve the first-quantized problem will then occupy us for the rest of the paper.

In first quantization, our auxiliary question can be interpreted as follows. We begin with the critical bosonic string theory in the flat uncompactified spacetime with coordinates X^I, $I = 1, \ldots 26$, described by the Polyakov action

$$W = \frac{1}{4\pi\alpha'} \int d^2\mathbf{x} \, \sqrt{g} \, g^{ij} \partial_i X^I \partial_j X^I. \tag{2.1}$$

Its partition function $\mathcal{F}_h$ on a compact worldsheet Σ_h of genus h is defined as the path integral

$$\mathcal{F}_h = \int_{\mathcal{A}_h/\mathcal{G}_h} \mathcal{D}X(\mathbf{x}) \, \mathcal{D}g_{ij}(\mathbf{x}) \, \exp\left\{-W[X(\mathbf{x}), g_{ij}(\mathbf{x})]\right\}, \tag{2.2}$$

where $\mathcal{A}_h$ is the space of all fields $X^I(\mathbf{x})$ and $g_{ij}(\mathbf{x})$ on Σ_h, the gauge group $\mathcal{G}_h$ consists of worldsheet diffeomorphisms and Weyl transformations of Σ_h, and $\mathcal{D}X \, \mathcal{D}g_{ij}$ schematically denotes the appropriate measure on the space of gauge orbits $\mathcal{A}_h/\mathcal{G}_h$. We would like to construct a $2 + 1$ dimensional quantum theory designed such that when it is quantized canonically on $\Sigma_h \times \mathbf{R}$, this theory has a ground state whose unnormalized wavefunction $|\Psi_{0h}\rangle$ reproduces the string partition function on Σ_h,

$$\mathcal{F}_h = \langle \Psi_{0h} | \Psi_{0h} \rangle. \tag{2.3}$$

More precisely, we will impose a stronger condition which will imply (2.3): Representing the ground state $|\Psi_{0h}\rangle$ in the Schrödinger representation as a functional $\Psi_{0h}[X(\mathbf{x}), g_{ij}(\mathbf{x})]$, we will require that it reproduces

$$\mathcal{D}X \, \mathcal{D}g_{ij} \, \exp\left\{-W[X, g_{ij}]\right\} = \Psi_{0h}^*[X, g_{ij}] \Psi_{0h}[X, g_{ij}], \tag{2.4}$$

as an equality between two densities on the space of gauge orbits $\mathcal{A}_h/\mathcal{G}_h$. The subsequent integral over $\mathcal{A}_h/\mathcal{G}_h$ then leads to (2.3).

In the case of $h = 0$, the partition function $\mathcal{F}_0$ of the critical bosonic string on S^2 vanishes indentically, because the measure in (2.2) contains the inverse volume of the noncompact conformal Killing symmetry group $\mathrm{SL}(2, \mathbf{C})$. This suggests that any membrane

theory which solves our first-quantized problem should have no normalizable ground-state wavefunction on $S^2 \times \mathbf{R}$.

In the rest of this section, we will assume that $|\Psi_{0h}\rangle$ which satisfies (2.3) is known, and show that under this assumption, the second-quantized problem can be solved by elementary methods of many body theory. We first define the second-quantized string partition function $\mathcal{Z}$ of the closed bosonic string theory to all orders in the string coupling g_s,

$$\mathcal{Z} \equiv \exp \left\{ \sum_{h=0}^{\infty} g_s^{2h-2} \mathcal{F}_h \right\}. \tag{2.5}$$

This expression has a well-defined limit as $g_s \to 0$, because $\mathcal{F}_0 = 0$ as mentioned above.

We wish to find a ground state of the second-quantized theory of membranes which reproduces this partition function $\mathcal{Z}$. This state will be an element of the Fock space $\mathfrak{H}$ of multi-membrane states. (We denote states in this second-quantized Hilbert space by $| \;\rangle\!\rangle$, to distinguish them from the first-quantized quantum states of a single membrane.) Thus, we are looking for a ground state $|\Psi_0\rangle\!\rangle$ which satisfies

$$\mathcal{Z} = \langle\!\langle \Psi_0 | \Psi_0 \rangle\!\rangle. \tag{2.6}$$

We choose to present the solution to the second-quantized problem first, because the answer is robust and rather insensitive to the precise form of the solution to the first-quantized problem. In fact, it is natural to expect that if the first-quantized problem has a solution, it will not be unique — two distinct theories on the membrane worldvolume can share the same ground state but differ in their spectra of excited states. This is analogous to the relationship between static and dynamical critical phenomena: A single static universality class can split into several distinct dynamical universality classes, which all share the same equilibrium properties.

2.1 The Fock space of membranes

Imagine that we have been given a basis of single-membrane quantum states,

$$|\Psi_{0h}\rangle, \qquad \{|\Psi_{\alpha h}\rangle\}, \tag{2.7}$$

where the index α denotes collectively all the excited states, and the ground state $|\Psi_{0h}\rangle$ satisfies (2.3). The second-quantized Hilbert space $\mathfrak{H}$ of multi-membrane states can then be constructed by elementrary methods of many-body physics. First we associate a pair of creation and annihilation operators with each state,

$$A_{0h}, A_{0h}^{\dagger}, \qquad A_{\alpha h}, A_{\alpha h}^{\dagger}. \tag{2.8}$$

These satisfy the canonical commutation relations,

$$[A_{0h}, A_{0h'}^{\dagger}] = \delta_{hh'}, \qquad [A_{\alpha h}, A_{\beta h'}^{\dagger}] = \delta_{\alpha\beta}\delta_{hh'}, \tag{2.9}$$

(with all the unlisted commutators equal to zero), and can be used to define the second-quantized Fock space $\mathfrak{H}_h$ of quantum membranes of genus h, by first defining the Fock space vacuum $|0\rangle\!\rangle_h$ via

$$A_{0h}|0\rangle\!\rangle_h = 0, \qquad A_{\alpha h}|0\rangle\!\rangle_h = 0. \tag{2.10}$$

The Fock space $\mathfrak{H}_h$ is then built in the standard way by the action of the creation operators $A_{0h}^\dagger$, $A_{\alpha h}^\dagger$ on $|0\rangle\!\rangle_h$. Each creation operator creates a membrane in the corresponding quantum state $|\Psi_{0h}\rangle$ or $|\Psi_{\alpha h}\rangle$. States in $\mathfrak{H}_h$ thus correspond to collections of an arbitrary number of quantum membranes, each of genus h.

The total Hilbert space $\mathfrak{H}$ is the tensor product of $\mathfrak{H}_h$ over all values of h,

$$\mathfrak{H} \equiv \bigotimes_{h=0}^\infty \mathfrak{H}_h, \tag{2.11}$$

The total Hilbert space $\mathfrak{H}$ is the Fock space generated by the application of arbitrary collections of creation operators on the Fock vacuum, defined as

$$|0\rangle\!\rangle \equiv \bigotimes_{h=0}^\infty |0\rangle\!\rangle_h. \tag{2.12}$$

2.2 Bose-Einstein condensation and spacetime superfluidity

The second-quantized Fock space $\mathfrak{H}$ contains multi-membrane states, with any number of membranes of any genera. We claim that our desired state $|\Psi_0\rangle\!\rangle$ is a specific state in $\mathfrak{H}$, given by

$$|\Psi_0\rangle\!\rangle = \bigotimes_{h=0}^\infty \exp\left\{ g_s^{h-1} A_{0h}^\dagger \right\} |0\rangle\!\rangle_h = \exp\left\{ \sum_h g_s^{h-1} A_{0h}^\dagger \right\} |0\rangle\!\rangle. \tag{2.13}$$

A direct calculation using (2.3) and the elementary algebra of creation and annihilation operators shows that (2.13) indeed satisfies the desired property (2.6).

This state has an interesting intuitive interpretation. For each value of h, this state looks like the ground state of a spacetime theory in which membranes of genus h — all in their ground state $|\Psi_{0h}\rangle$ — have formed a spacetime Bose-Einstein condensate. Defining the number operators in each membrane sector via

$$N_{0h} = A_{0h}^\dagger A_{0h}, \qquad N_{\alpha h} = A_{\alpha h}^\dagger A_{\alpha h}, \tag{2.14}$$

we see that the proposed ground state (2.13) is not an eigenstate of N_{0h} and thus does not contain a definite finite number of membranes.

Instead, the ground state is an eigenstate of A_{0h}, with eigenvalue g_s^{h-1}. This indicates that the strength of the condensate of membranes of different genera is correlated over all h, via the value of the string coupling constant g_s. It would be interesting to study model Hamiltonians that reproduce the same ground state. Such an analysis would be difficult in the absence of at least some information about the membrane excited states. However, the correlated nature of the condensate suggests that in this second-quantized ground state, membranes interact via a local contact interaction, which is insensitive to the global topology of the membranes. This interaction allows processes in which a membrane of genus h gets pinched into a pair of genera h' and $h - h'$ respectively, or the self-pinching in which the genus changes to $h - 1$, plus the reversal of these two processes.

In conventional condensed matter systems, ground states that take the form of a Bose-Einstein condensate typically lead to superfluidity, characterized by gapless excitations

with an emergent relativistic low-energy dispersion relation. It is tempting to predict that the membrane theory whose ground state is given by (2.13) similarly exhibits spacetime superfluidity. However, the knowledge of the ground state (2.13) itself is not sufficient to determine whether the system has excitations that behave as those of a superfluid. As pointed out above, it is possible that different solutions of the first-quantized problem might exist, sharing the same single-membrane ground state $|\Psi_{0h}\rangle$ but with different spectra of excited states $|\Psi_{\alpha h}\rangle$. For example, a minimal solution to our first-quantized problem would be given by a theory of membranes where the ground state $|\Psi_{0h}\rangle$ is the only physical state on Σ_h, and the membrane has no physical excited states (and in particular, no states carrying nonzero values of spatial momenta). In such a minimal realization, the spacetime theory would be effectively a topological theory, and the Bose-Einstein condensate would not be accompanied by spacetime superfluidity.

3 The first-quantized theory

Having shown how the second-quantized problem is solved assuming the existence of a worldsvolume theory that reproduces the worldsheet path integral of string theory genus by genus, it now remains to solve the corresponding first-quantized problem.

3.1 Worldvolume matter: Lifshitz scalars and quantum criticality

We will pose the question first for a single worldsheet scalar field, before coupling to worldsheet gravity. In fact, it will be useful to take the broader perspective and consider a scalar field theory in D flat Euclidean dimensions $\mathbf{x} = (x^i)$, $i = 1, \ldots D$, with the Euclidean action

$$W = \frac{1}{2} \int d^D\mathbf{x} \, (\partial_i \Phi \partial_i \Phi). \tag{3.1}$$

The partition function of the free scalar takes the form

$$\mathcal{Z} = \int \mathcal{D}\Phi(\mathbf{x}) \exp\left\{-W[\Phi(\mathbf{x})]\right\}, \tag{3.2}$$

of a path integral on the space of field configurations $\Phi(\mathbf{x})$. Imagine now a theory in $D + 1$ dimensions whose configuration space coincides with the space of all $\Phi(\mathbf{x})$. In the Schrödinger representation, the wavefunctions of this theory are functionals $\Psi[\Phi(\mathbf{x})]$. For any given wavefunction, $|\Psi[\Phi(\mathbf{x})]|^2$ is naturally a density on the configuration space. We want to design our system such that its ground-state wavefunction $\Psi_0[\Phi(\mathbf{x})]$ reproduces the path integral density of (3.2),

$$\mathcal{D}\Phi(\mathbf{x}) \exp\left\{-W[\Phi(\mathbf{x})]\right\} = \Psi_0^*[\Phi(\mathbf{x})] \, \Psi_0[\Phi(\mathbf{x})]. \tag{3.3}$$

As it turns out, a construction which yields the desired answer for the scalar field is known in the condensed matter literature (see, e.g., [9]). It is given in terms of a slightly exotic scalar field theory in $D + 1$ dimensions, whose action is

$$S = \frac{1}{2} \int dt \, d^D\mathbf{x} \left\{ (\dot{\Phi})^2 - \frac{1}{4}(\Delta\Phi)^2 \right\}. \tag{3.4}$$

Here Δ is the spatial Laplacian, $\Delta = \partial_i \partial_i$. Note that S is a sum of a "kinetic term" involving time derivatives, and a "potential term" which is of a special form: It can be derived from a variational principle,

$$\frac{1}{4}(\Delta\Phi(\mathbf{x}))^2 = \left(\frac{1}{2}\frac{\delta W}{\delta\Phi(\mathbf{x})}\right)^2, \tag{3.5}$$

where W is the action (3.1) of the Euclidean scalar theory. Henceforth we say that theories that enjoy this property satisfy the "detailed balance" condition. This property, and its extension to the case involving gravity, will play a central role in the rest of the paper.

The scalar field theory (3.4) is a prototype of a class of models introduced and studied in the context of tri-critical phenomena in condensed matter physics by Lifshitz [10] in 1941, and is consequently referred to in the literature as the "Lifshitz scalar" field theory [11, 12]. In the context originally studied by Lifshitz [10], t is Wick rotated to become one of the spatial dimensions, and the Lifshitz scalar then describes the tricritical point connecting the phases with a zero, homogeneous or spatially modulated condensate of Φ. The same theory is also relevant in the description of various universality classes in dynamical critical phenomena, and in quantum criticality. In particular, the Lifshitz scalar is believed to be in the same universality class as the quantum dimer problem, which is particularly intriguing because of the close connection [8, 13] between dimer models, topological string theory and noncritical M-theory.

Why is the ground-state wavefuction of the Lifshitz scalar theory related to the partition function (3.2)? The key to this fact is the detailed balance condition obeyed by the Lifshitz scalar. To identify the ground-state wavefunction, we quantize the theory canonically. The Hamiltonian of the Lifshitz scalar is

$$H = \frac{1}{2}\int d^D\mathbf{x}\left\{P^2 + \frac{1}{4}(\Delta\Phi)^2\right\}. \tag{3.6}$$

In the Schrödinger representation we realize the momenta conjugate to $\Phi(\mathbf{x})$ as

$$P(\mathbf{x}) = -i\frac{\delta}{\delta\Phi(\mathbf{x})}. \tag{3.7}$$

Up to a normal-ordering constant, the Hamiltonian can be written as

$$H = \frac{1}{2}\int d^D\mathbf{x}\left(-\frac{\delta}{\delta\Phi} - \frac{1}{2}\Delta\Phi\right)\left(\frac{\delta}{\delta\Phi} - \frac{1}{2}\Delta\Phi\right) = \int d^D\mathbf{x}\,\overline{\mathcal{Q}}\mathcal{Q}, \tag{3.8}$$

where

$$\mathcal{Q}(\mathbf{x}) = iP(\mathbf{x}) - \frac{1}{2}\Delta\Phi(\mathbf{x}) = \frac{\delta}{\delta\Phi(\mathbf{x})} - \frac{1}{2}\Delta\Phi(\mathbf{x}), \tag{3.9}$$

and $\overline{\mathcal{Q}}$ is its complex conjugate. Consequently, any functional $\Psi_0[\Phi(\mathbf{x})]$ that is annihilated by $\mathcal{Q}$,

$$\mathcal{Q}\Psi_0[\Phi(\mathbf{x})] \equiv \left(\frac{\delta}{\delta\Phi(\mathbf{x})} + \frac{1}{2}\Delta\Phi(\mathbf{x})\right)\Psi_0[\Phi(\mathbf{x})] = 0, \tag{3.10}$$

is an eigenstate of the Hamiltonian with the lowest eigenvalue, and thus represents a candidate wavefunction of the ground state. In order for this candidate to be a true wavefunction, it must be normalizable.

Because the Lifshitz scalar theory satisfies the detailed balance condition, it is easy to find a simple solution to (3.10), given by

$$\Psi_0[\Phi(\mathbf{x})] = \exp\left\{ -\frac{1}{4} \int d^D\mathbf{x}\, \partial_i\Phi\partial_i\Phi \right\}. \tag{3.11}$$

This is a normalizable wavefunction of the ground state, which in turn yields (3.3) and solves our problem.

The fact that the Hamiltonian (3.8) can be written as $\int \overline{\mathcal{Q}}\mathcal{Q}$, and the subsequent role played by the simpler condition $\mathcal{Q}\Psi_0 = 0$ in indentifying the lowest energy eigenstates, are reminiscent of supersymmetry and the role played by the BPS condition. This resemblance is not accidental, and can be rephrased in terms of an underlying supersymmetry with scalar supercharges, formally similar to topological BRST symmetry. In the context of condensed matter applications mentioned above, this symmetry is known as the Parisi-Sourlas supersymmetry [14] (see also [15] for a nice early review). In the context of strings and membranes, Parisi-Sourlas supersymmetry played a role in [16]. We will not use the supersymmetric formalism in the present paper.

Regardless of its relation with the Euclidean scalar theory in D dimensions, the Lifshitz scalar theory in $D+1$ dimensions is an interesting system in its own right. Its action (3.4) defines a Gaussian RG fixed point, with scaling properties which are somewhat exotic from the perspective of relativistic quantum field theory. We will measure the scaling properties of various quantities in the units of inverse spatial length. In order for the two terms in the action to scale the same way, we must assign anisotropic scaling properties to space and time,

$$[\mathbf{x}] = -1, \qquad [t] = -2. \tag{3.12}$$

In condensed matter systems, the degree of anisotropy between time and space is measured by the dynamical cricial exponent z. Lorentz symmetry in relativistic systems implies $z = 1$, while nonrelativistic systems with Galilean invariance have $z = 2$. More generally, the dynamical critical exponent can be defined in terms of the scaling properties of two-point functions,

$$\langle\Phi(\mathbf{x},t)\Phi(0,0)\rangle = \frac{1}{|\mathbf{x}|^{2[\Phi]}}\, f\left(\frac{\mathbf{x}}{t^{1/z}}\right), \tag{3.13}$$

where $[\Phi]$ is the conformal dimension of Φ. In the case of the free Lifshitz scalar theory, we have $z = 2$, and

$$[\Phi] = \frac{D-2}{2}. \tag{3.14}$$

This conformal dimension is of course different from the dimension $[\Phi]_{z=1}$ of the scalar field at the relativistic Gaussian fixed point in $D+1$ dimensions, which is $[\Phi]_{z=1} = (D-1)/2$. As a result, the lower critical dimension of the Lifshitz scalar at which the two-point function

becomes logarithmic is $2+1$, and not $1+1$ as in the usual relativistic case.[1] Remarkably, making the system anisotropic causes a shift in the critical dimension of the system.

In the case of relativistic scalar field theory, the importance of $1+1$ being the critical dimension can hardly be overstated. This fact is at the core of string theory, and represents perhaps the most elegant way [17, 18] of deriving Einstein's equations and their systematic higher-order corrections, from the simple condition of quantum conformal invariance of the nonlinear sigma model. Similarly, one can can generalize the Lifshitz scalar theory to an anisotropic nonlinear sigma model, which will have an infinite number of classically marginal couplings in $2+1$ dimensions. A detailed study of the RG properties of such Lifshitz-type sigma models should be very interesting.

3.2 Requirements on worldvolume gravity

In order to extend the construction from the matter sector to the full string worldsheet theory, we need to couple the Lifshitz scalar theory to some form of worldvolume gravity. When this worldvolume system is quantized on $\Sigma_h \times \mathbf{R}$, the resulting wavefunction of the membrane ground state is supposed to reproduce (2.4). Consequently, the ground-state wavefunction must be a functional of $X^I(\mathbf{x})$ and $g_{ij}(\mathbf{x})$ defined on the space of gauge orbits $\mathcal{A}_h/\mathcal{G}_h$: In other words, Ψ_0 must be invariant under worldsheet diffeomorphisms and Weyl transformations.

Our task is to design a gravity theory in $2+1$ dimensions which reproduces these expected properties of the ground-state wavefunction, much like the Lifshitz scalar reproduces the path integral of the worldsheet matter sector. This gravity theory should naturally couple to the anisotropic theory of matter described by Lifshitz scalars with $z = 2$. In order to match the scaling properties of the matter sector, this gravity theory should therfore also be at quantum criticality with $z = 2$.

The possibility of constructing a nonrelativistic theory of gravity with anisotropic scaling and nontrivial values of z is clearly of a more general interest. Therefore, we devote section 4 to the presentation of such anisotropic gravity models in the general case of $D+1$ dimensions, and return to $D = 2$ in section 5.

4 Gravity at a $z = 2$ Lifshitz point in $D + 1$ dimensions

In this section, we formulate a classical theory of gravity with dynamical critical exponent $z = 2$. As in the case of the Lifshitz scalar reviewed in section 3.1, it will be instructive to consider our construction in $D + 1$ dimensions, specializing to the case of $D = 2$ only later as required for the application to the membrane worldvolume.

We will assume that our spacetime is topologically of the form $\mathcal{M} = \mathbf{R} \times \Sigma$, where Σ is a compact D-dimensional space. This assumption will simplify our construction, by avoiding the discussion of the possible spatial boundary terms in the action.

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[1]Straightforward generalizations of the Lifshitz theory exist [12], such as theories at the (m, n) Lifshitz point, with m dimensions like t and n dimensions like $\mathbf{x}$.

4.1 First ingredients

As a minimal requirement, our theory in $D + 1$ dimensions should describe spatial components $g_{ij}(\mathbf{x}, t)$ of the metric, $i, j = 1, \ldots D$. The gauge symmetries will surely have to contain diffeomorphisms of space. Motivated by the form of the Lagrangian for the Lifshitz scalar, our gravity theory will have a kinetic term given by

$$S_K = \frac{1}{2\kappa^2} \int dt \, d^D\mathbf{x} \sqrt{g} \, \dot{g}_{ij} \, G^{ijk\ell} \, \dot{g}_{k\ell}. \tag{4.1}$$

We have introduced a coupling constant κ, whose physical role will become clear later, in section 4.5. Throughout the paper, we use "˙" to denote the time derivative; e.g., $\partial_t g_{ij} \equiv \dot{g}_{ij}$.

In order to write down this kinetic term, we needed a "metric on the space of metrics," denoted here by $G^{ijk\ell}$. Spatial diffeomorphism invariance of the action requires $G^{ijk\ell}$ to take, up to an overall normalization, the following form

$$G^{ijk\ell} = \frac{1}{2} \left(g^{ik} g^{j\ell} + g^{i\ell} g^{jk} \right) - \lambda \, g^{ij} g^{k\ell}, \tag{4.2}$$

with λ an arbitrary real constant. This object is very similar to the familiar De Witt metric of general relativity. In the relativistic theory, the full spacetime diffeomorphism invariance fixes the value of λ uniquely, to equal $\lambda = 1$. In that case, the "metric on the space of metrics" $G^{ijk\ell}$ is known as the "De Witt metric." We will extend this terminology to our more general case as well, even when λ is not necessarily equal to one.

For now, λ plays the role of a coupling constant. In section 5 we will see that the value of this coupling is uniquely determined, if we require that the theory also respect an anisotropic version of local Weyl invariance.

4.2 The potential term

In our gravity theory, we wish to maintain the anisotropy of scaling between space and time, consistent with the value of dynamical critical exponent $z = 2$ of the Lifshitz matter fields. As a result, we look for a "potential term" S_V of fourth-order in spatial derivatives, so that the full action of our system is the sum of two parts,

$$S = S_K - S_V. \tag{4.3}$$

In principle, if we follow the usual logic of effective field theory, many such terms can be written down and therefore should be included in the effective action. However, the choices can be severely reduced, if we require the existence of an action $W[g_{ij}(\mathbf{x})]$ in D dimensions such that

$$S_V = \frac{\kappa^2}{2} \int dt \, d^D\mathbf{x} \sqrt{g} \, E^{ij} \, \mathcal{G}_{ijk\ell} \, E^{k\ell}, \tag{4.4}$$

where E^{ij} follows from the variational principle with action $W[g_{ij}(\mathbf{x})]$,

$$\sqrt{g} \, E^{ij} = \frac{1}{2} \frac{\delta W}{\delta g_{ij}}. \tag{4.5}$$

In other words, we require that our potential term satisfies the gravitational analog of the "detailed balance" condition mentioned in section 3.1.

In (4.4), $\mathcal{G}_{ijk\ell}$ denotes the inverse of the De Witt metric,

$$\mathcal{G}_{ijmn}G^{mnk\ell} = \frac{1}{2}\left(\delta_i^k\delta_j^\ell + \delta_i^\ell\delta_j^k\right). \tag{4.6}$$

More explicitly, we have

$$\mathcal{G}_{ijk\ell} = \frac{1}{2}\left(g_{ik}g_{j\ell} + g_{i\ell}g_{jk}\right) - \widetilde{\lambda}\,g_{ij}g_{k\ell}, \tag{4.7}$$

with

$$\widetilde{\lambda} = \frac{\lambda}{D\lambda - 1}. \tag{4.8}$$

In order to end up with S_V which is invariant under spatial diffeomorphisms and of fourth order in spatial derivatives, we must take W to be the Einstein-Hilbert action in D dimensions,[2]

$$W = \frac{1}{\kappa_W^2}\int d^D\mathbf{x}\,\sqrt{g}\,R. \tag{4.9}$$

The general action W could also contain a cosmological constant term Λ_W. However, for now, we set $\Lambda_W = 0$ in order to focus on the leading term in W that produces the dominant, highest-dimension operators in S_V. We will return to the discussion of the general case with nonzero Λ_W in section 4.6.

With the choice of the Einstein-Hilbert term (4.9) as W, the full action is given by

$$\begin{aligned}
S &= \frac{1}{2}\int dt\,d^D\mathbf{x}\,\sqrt{g}\left\{\frac{1}{\kappa^2}\dot{g}_{ij}G^{ijk\ell}\dot{g}_{k\ell} - \frac{\kappa^2}{4\kappa_W^4}\left(R^{ij} - \frac{1}{2}Rg^{ij}\right)\mathcal{G}_{ijk\ell}\left(R^{k\ell} - \frac{1}{2}Rg^{k\ell}\right)\right\} \\
&= \frac{1}{2}\int dt\,d^D\mathbf{x}\,\sqrt{g}\left\{\frac{1}{\kappa^2}\dot{g}_{ij}G^{ijk\ell}\dot{g}_{k\ell} - \frac{\kappa^2}{4\kappa_W^4}\left(R^{ij}R_{ij} + aR^2\right)\right\},
\end{aligned} \tag{4.10}$$

where a is a constant equal to

$$a = \frac{1 - \lambda - D/4}{D\lambda - 1}. \tag{4.11}$$

Note that in a large range of values of D and λ the potential term S_V in the action is manifestly positive definite.

4.3 Extending the gauge symmetries

The action in (4.10) appears to be a good first step, but it is only invariant under *spatial* diffeomorphisms

$$\delta x^i = \zeta^i(x^j) \tag{4.12}$$

and global time translations, and there is no Weyl invariance. As a result, when we specialize to $D = 2$, the hypothetical ground-state wavefunction would depend on the conformal factor of the two-dimensional metric. Thus, we will need to accomodate Weyl

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[2]Our notation in this paper is strictly nonrelativistic: All quantities such as the covariant derivative ∇_i, the Ricci scalar R, the Ricci tensor R_{ij} *etc.*, are always defined in terms of the metric g_{ij} on the D-dimensional leaves of the spacetime foliation, unless explicitly stated otherwise.

invariance in order to make contact with the partition function of critical string theory. The way to resolve these issues is to require extended gauge symmetries, which will in turn require new gauge fields.

4.3.1 Foliation-preserving diffeomorphisms

Given the preferred role of time in our theory, it is natural to extend the gauge symmetry of time-independent spatial diffeomorphisms enjoyed by (4.10) to all spacetime diffeomorphisms that respect the preferred codimension-one foliation[3] $\mathcal{F}$ of spacetime $\mathcal{M}$ by the slices of fixed time. Such "foliation-preserving diffeomorphisms" will consist of spacetime-dependent spatial diffeomorphisms as well as time-dependent time reparametrizations, generated by infinitesimal transformations

$$\delta x^i = \zeta^i(t, \mathbf{x}), \qquad \delta t = f(t). \tag{4.13}$$

Together with the new symmetries, we also introduce new fields, N and N_i. From the point of view of the $D + 1$ canonical ADM formalism in relativistic gravity, these are the well-known "lapse and shift" variables. Thus, our theory will share its field content with conventional relativistic gravity theory, at least if N and N_i are allowed to be functions of both space and time.

4.3.2 Spacetime diffeomorphisms and the nonrelativistic limit

The algebra of foliation-preserving diffeomorphisms and its action on the fields g_{ij}, N_i and N can be conveniently derived from the relativistic action of all diffeomorphisms on $g_{\mu\nu}$, by restoring the speed of light c and taking the nonrelativistic limit $c \to \infty$. We start with the relativistic metric $g_{\mu\nu}$ in the usual ADM decomposition but with c restored,

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + N_i N^i/c^2, & N_i/c \\ N_i/c, & g_{ij} \end{pmatrix}, \tag{4.14}$$

and with $x^0 = ct$. One can view this expression as a leading order of an expansion in $1/c$. Similarly, we expand the generators v^μ of spacetime diffeomorphisms in the powers of $1/c$,

$$v^t = cf(t, \mathbf{x}) + \mathcal{O}(1/c), \qquad v^i = \zeta^i(t, \mathbf{x}) + \mathcal{O}(1/c^2). \tag{4.15}$$

In order to obtain a nonsingular $c \to \infty$ limit, the generator of time reparametrizations f in (4.15) must be restricted to be a function of t only. With this condition, the standard action of relativistic diffeomorphism generators v^μ on $g_{\mu\nu}$ contracts to the diffeomorphisms (4.13) that preserve the preferred foliation of spacetime by leaves of constant time t. Their action on the component fields is obtained by taking the $c \to \infty$ limit of the

---

[3]A codimension-q foliation $\mathcal{F}$ of a d-dimensional manifold $\mathcal{M}$ is defined as $\mathcal{M}$ equipped with an atlas of coordinate systems (y^a, x^i) $a = 1, \ldots q$, $i = 1, \ldots d - q$, such that the transition functions take the restricted form $(\widetilde{y}^a, \widetilde{x}^i) = (\widetilde{y}^a(y^b), \widetilde{x}^i(y^b, x^j))$. For the general theory of foliations, see e.g. [19–21] and references therein.

relativistic diffeomorphisms v^μ acting on $g_{\mu\nu}$, which leads to

$$
\begin{aligned}
\delta g_{ij} &= \partial_i \zeta^k g_{jk} + \partial_j \zeta^k g_{ik} + \zeta^k \partial_k g_{ij} + f\, \dot{g}_{ij}, \\
\delta N_i &= \partial_i \zeta^j N_j + \zeta^j \partial_j N_i + \dot{\zeta}^j g_{ij} + \dot{f}\, N_i + f\, \dot{N}_i, \\
\delta N &= \zeta^j \partial_j N + \dot{f}\, N + f\, \dot{N}.
\end{aligned}
\tag{4.16}
$$

In order to obtain a smooth $c \to \infty$ limit, f can only be a function of t, while ζ^i is allowed to depend on both t and x^j: The algebra becomes that of the foliation-preserving diffeomorphisms (4.13). Note that the transformation rules under foliation-preserving diffeomorphisms do not depend of the anticipated value of the dynamical exponent z that measures the degree of anisotropy between space and time. Thus, the value of z is not determined by the gauge symmetries, and represents an interesting dynamical quantity in our theory.

Because the generator of time diffeomophisms $f(t)$ is a function of time only, the gauge symmetry of foliation-preserving diffeomorphisms has one less generator per spacetime point than general diffeomorphism symmetry. It is natural to match this by restricting the corresponding gauge field N, associated with the time diffeomorphisms, to also be a function of only t. This step is not strictly mandated by the structure of the symmetry transformations (4.16), but allowing N to be a general function of t and $\mathbf{x}$ would lead to difficulties in quantization, at least in the absence of extra gauge symmetries.

There is an interesting possibility of taking the nonrelativistic limit in such a way that the number of local symmetries matches that of general relativity. It involves keeping the subleading term in the $1/c$ expansion of the time-time component of the metric,[4] $g_{00} = -N^2 + (N_i N^i + 2A)/c^2$, and keeping the subleading term in the time component of the diffeomorphism transformation, $v^t = cf(t) - \varepsilon(t, \mathbf{x})/c$. It turns out that $\varepsilon(t, \mathbf{x})$ acts on the fields by

$$
\begin{aligned}
\delta_\varepsilon A &= N^2 \dot{\varepsilon} + N \dot{N} \varepsilon - N^2 N^i \partial_i \varepsilon, \\
\delta_\varepsilon N_i &= N^2 \partial_i \varepsilon, \\
\delta_\varepsilon N &= \delta_\varepsilon g_{ij} = 0.
\end{aligned}
\tag{4.17}
$$

If the leading term $N(t)$ in g_{00} is restricted to be only a function of time as suggested above, this new symmetry is simply an Abelian gauge symmetry with gauge parameter $N\varepsilon$, and with A/N and N_i/N transforming as an Abelian connection. However, extending the foliation-preserving diffeomorphisms by this Abelian gauge symmetry appears to run into difficulties with constructing nontrivial Lagrangians invariant under this symmetry, and we will not pursue the possibility of such an extended gauge symmetry in this paper.

4.3.3 The covariant action

In order to make our theory invariant under foliation-preserving diffeomorphisms, we need to decorate various terms in the action (4.10) by the appropriate dependence on N and N_i.

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[4]This is the place which would be occupied in the nonrelativistic expansion of general relativity by the Newton potential.

For example, the covariant volume element is $\sqrt{g}N$, and the time derivative of the metric is replaced by

$$\dot{g}_{ij} \to \frac{1}{N} \left(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i \right), \tag{4.18}$$

which transforms covariantly under foliation-preserving diffeomorphisms. Similarly, any term of the form

$$\int dt\, d^D\mathbf{x}\, \sqrt{g}\, V[g_{ij}] \tag{4.19}$$

which respects time-independent spatial diffeomorphisms and does not depend on time derivatives of g_{ij} can be covariantized as

$$\int dt\, d^D\mathbf{x}\, \sqrt{g}\, N\, V[g_{ij}]. \tag{4.20}$$

Indeed, we have

$$\delta_f \int dt\, d^D\mathbf{x}\, \sqrt{g}\, N\, V[g_{ij}] = \int dt\, d^D\mathbf{x}\, \partial_t \left(\sqrt{g}\, f\, N\, V[g_{ij}] \right). \tag{4.21}$$

In the end, the full covariant action is

$$S = \frac{1}{2} \int dt\, d^D\mathbf{x}\, \sqrt{g} \left\{ \frac{1}{\kappa^2 N} \left(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i \right) G^{ijk\ell} \left(\dot{g}_{k\ell} - \nabla_k N_\ell - \nabla_\ell N_k \right) \right.$$
$$\left. - \frac{\kappa^2 N}{4\kappa_W^4} \left(R^{ij} - \frac{1}{2} R g^{ij} \right) \mathcal{G}_{ijk\ell} \left(R^{k\ell} - \frac{1}{2} R g^{k\ell} \right) \right\}. \tag{4.22}$$

Setting $N = 1$, $N_i = 0$ would restore the reduced action (4.10).

4.3.4 Detailed balance condition

As is the case for relativistic quantum field theories, explicit calculations are most conveniently performed after the Wick rotation to imaginary time, $\tau = it$. This rotation entails $N_j \to iN_j$. After the rotation, the action can be rewritten — up to total derivatives — as a sum of squares,[5]

$$S = \frac{i}{2} \int d\tau\, d^D\mathbf{x}\, \sqrt{g} N \left\{ \left[\frac{1}{\kappa N} \left(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i \right) + \frac{\kappa}{2\kappa_W^2} \mathcal{G}_{ijmn} \left(R^{mn} - \frac{1}{2} R g^{mn} \right) \right] \right.$$
$$\left. \times G^{ijk\ell} \left[\frac{1}{\kappa N} \left(\dot{g}_{k\ell} - \nabla_k N_\ell - \nabla_\ell N_k \right) + \frac{\kappa}{2\kappa_W^2} \mathcal{G}_{k\ell pq} \left(R^{pq} - \frac{1}{2} R g^{pq} \right) \right] \right\}. \tag{4.23}$$

In order to see that (4.22) is indeed reproduced from (4.23) by the inverse Wick rotation, we need to show that the cross-terms

$$\int d\tau\, d^D\mathbf{x}\, \sqrt{g} N \left\{ \left[\frac{1}{\kappa N} \left(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i \right) \right] G^{ijk\ell} \left[\frac{\kappa}{2\kappa_W^2} \mathcal{G}_{k\ell pq} \left(R^{pq} - \frac{1}{2} R g^{pq} \right) \right] \right\} \tag{4.24}$$

are a sum of total derivatives. First, we have

$$\int d\tau\, d^D\mathbf{x}\, \sqrt{g} N \left\{ \frac{1}{\kappa N} \dot{g}_{ij} G^{ijk\ell} \left[\frac{\kappa}{2\kappa_W^2} \mathcal{G}_{k\ell pq} \left(R^{pq} - \frac{1}{2} R g^{pq} \right) \right] \right\}$$
$$= -\frac{1}{2} \int d\tau\, d^D\mathbf{x}\, \dot{g}_{ij} \frac{\delta W}{\delta g_{ij}} = -\frac{1}{2} \int d\tau\, d^D\mathbf{x}\, \partial_\tau (\mathcal{L}_W), \tag{4.25}$$

---

[5]In the Wick-rotated theory, "˙" denotes differentiation with respect to the imaginary time τ.

where $\mathcal{L}_W$ is the Lagrangian density, $W = \int d\tau \, d^D\mathbf{x} \, \mathcal{L}_W$. For this to hold, it was crucial that (i) the potential term S_V is a square of terms (4.5) which originate from a variational principle, and (ii) that the metric $\mathcal{G}_{ijk\ell}$ used in the potential term S_V is the inverse of the De Witt metric $G^{ijk\ell}$ that appeared in the kinetic term S_K.

Similarly,

$$
\int d\tau \, d^D\mathbf{x} \, \sqrt{g} N \left[\frac{1}{\kappa N} \left(\nabla_i N_j + \nabla_j N_i \right) \right] G^{ijk\ell} \left[\frac{\kappa}{2\kappa_W^2} \mathcal{G}_{k\ell pq} \left(R^{pq} - \frac{1}{2} R g^{pq} \right) \right]
$$
$$
= -\int d\tau \, d^D\mathbf{x} \, \nabla_i N_j \frac{\delta W}{\delta g_{ij}} = -\int d\tau \, d^D\mathbf{x} \, \partial_i \left(N_j \frac{\delta W}{\delta g_{ij}} \right), \tag{4.26}
$$

as a consequence of the Bianchi identity $\nabla_i(R^{ij} - R g^{ij}/2) = 0$, or alternatively as a consequence of the gauge invariance of W under spatial diffeomorphisms.

Introducing an auxiliary field B^{ij}, we can rewrite (4.23) in the following form,

$$
S = \frac{i}{\kappa^2} \int d\tau \, d^D\mathbf{x} \, \sqrt{g} N \left\{ B^{ij} \left[\frac{1}{N} \left(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i \right) + \frac{\kappa^2}{2\kappa_W^2} \mathcal{G}_{ijk\ell} \left(R^{k\ell} - \frac{1}{2} R g^{k\ell} \right) \right] \right.
$$
$$
\left. - \frac{1}{2} B^{ij} \mathcal{G}_{ijk\ell} B^{k\ell} \right\}. \tag{4.27}
$$

All terms in (4.27) are at least linear in B^{ij}. This is a hallmark of similar constructions in nonequilibrium dynamics [22], dynamical critical phenomena [23, 24], quantum critical systems [9, 25] and stochastic quantization [26–28]. Moreover, the coefficient of the term linear in B^{ij} has a special form, intimately related to an evolution equation for g_{ij},

$$
\dot{g}_{ij} = -\frac{\kappa^2}{2\kappa_W^2} N \mathcal{G}_{ijk\ell} \left(R^{k\ell} - \frac{1}{2} R g^{k\ell} \right) + \nabla_i N_j + \nabla_j N_i \dot{g}_{ij}
$$
$$
\equiv \frac{\kappa^2}{2} N \mathcal{G}_{ijk\ell} \frac{\delta W}{\delta g_{k\ell}} + \nabla_i N_j + \nabla_j N_i. \tag{4.28}
$$

Since the curvature terms in this equation originated from the variational principle, this equation simply states that the evolution of g_{ij} is governed by a gradient flow $\delta W / \delta g_{ij}$ on the space of metrics, up to possible gauge transformations represented by N_i and N. In the context of condensed matter applications mentioned above, systems whose action S is so associated with a gradient flow generated by some W are said to satisfy the condition of "detailed balance." Investigating under what circumstances quantum corrections preserve these features of the action is the key to proving renormalizability of this setup.

Under rather general circumstances, theories which satisfy the detailed balance condition have simpler quantum properties than a generic theory in $D + 1$ dimensions. Their renormalization properties are often inherited from the simpler renormalization of the associated theory in D dimensions with action W, plus the possible renormalization of the relative normalization between the kinetic and potential terms in S. Examples of this phenomenon include scalar fields [29] or Yang-Mills gauge theories [28] (see also [2]). It will be important to analyze under what circumstances an analog of such "quantum inheritance principle" is valid for our nonrelativistic gravity models. This analysis is, however, beyond the scope of the present paper.

In passing, we note that the structure of the evolution equation (4.28) suggests an intimate relation between our theory of nonrelativistic gravity and the theory of Ricci flows, which in turn play a central role in Perelman's approach [30] to the Poincaré conjecture. Indeed, (4.28) is a covariantized Ricci flow equation, or more precisely a family of generalized Ricci flows parametrized by λ,

$$\dot{g}_{ij} = -\frac{\kappa^2}{2\kappa_W^2} N \left[R_{ij} + \frac{1-2\lambda}{2(D\lambda-1)} R g_{ij} \right] + \nabla_i N_j + \nabla_j N_i. \tag{4.29}$$

Setting $N = 1$ and $N_i = 0$ recovers the naive Ricci flow equation. The decorations of the naive flow in (4.29) by N and N_i take into account the fact that geometrically, we only care about the flow up to a – possibly time-dependent — spatial diffeomorphism and a time reparametrization. These gauge symmetries of the Ricci flow problem match naturally the foliation-preserving diffeomorphism invariance of our gravity theory.

4.4 Hamiltonian formulation

It is instructive to rewrite our theory with foliation-preserving diffeomorphism invariance in the canonical formalism, generalizing the ADM formulation of general relativity. The Hamiltonian formulation is particularly natural for the class of gravity theories proposed here, because the $D + 1$ split of the spacetime variables is naturally compatible with the preferred role of time and the anisotropic scaling.

The canonical momenta conjugate to g_{ij} are

$$\pi^{ij} = \frac{\delta S}{\delta \dot{g}_{ij}} = \frac{\sqrt{g}}{\kappa^2 N} G^{ijk\ell} \left(\dot{g}_{k\ell} - \nabla_k N_\ell - \nabla_\ell N_k \right) = \frac{2\sqrt{g}}{\kappa^2} G^{ijk\ell} K_{k\ell}, \tag{4.30}$$

where

$$K_{ij} = \frac{1}{2N} \left(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i \right) \tag{4.31}$$

is the extrinsic curvature tensor on the spatial leaves of the spacetime foliation. The momenta conjugate to N and N_i are identically zero, and their vanishing represent primary first-class constraints. The Poisson bracket of the canonical variables is

$$[g_{ij}(\mathbf{x}), \pi^{k\ell}(\mathbf{y})] = \frac{1}{2} \left(\delta_i^k \delta_j^\ell + \delta_i^\ell \delta_j^k \right) \delta^D(\mathbf{x} - \mathbf{y}). \tag{4.32}$$

In terms of the canonical variables, the Hamiltonian takes the form

$$H = \int d^D\mathbf{x} \left(N\mathcal{H} + N_i \mathcal{H}^i \right), \tag{4.33}$$

with $\mathcal{H}$ and $\mathcal{H}^i$ given by

$$\begin{aligned}
\mathcal{H} &= \frac{\kappa^2}{2\sqrt{g}} \pi^{ij} \mathcal{G}_{ijk\ell} \pi^{k\ell} + \frac{\kappa^2 \sqrt{g}}{4\kappa_W^4} \left(R^{ij} - \frac{1}{2} R g^{ij} \right) \mathcal{G}_{ijk\ell} \left(R^{k\ell} - \frac{1}{2} R g^{k\ell} \right) \\
&= \frac{\kappa^2}{2\sqrt{g}} \left[\pi^{ij} \pi_{ij} - \frac{\lambda}{D\lambda - 1} (\pi_i^i)^2 \right] + \frac{\kappa^2 \sqrt{g}}{4\kappa_W^4} \left(R^{ij} R_{ij} + a R^2 \right)
\end{aligned} \tag{4.34}$$

(where a is again as in (4.11)), and

$$\mathcal{H}^i = -2\nabla_j \pi^{ij}. \tag{4.35}$$

We would now like to calculate the algebra satisfied by the constraints in our nonrelativistic theory.

For comparison, it will be useful to recall first the structure of the relativistic constraints. In general relativity formulated in the canonical ADM formalism, the Hamiltonian is also given by (4.33). The momentum constraints $\mathcal{H}_i$ take the same form as given in (4.35), while $\mathcal{H}$ is replaced by the relativistic Hamiltonian constraint

$$\mathcal{H}_\perp = \frac{16\pi G_N}{2\sqrt{g}}\pi^{ij}\mathcal{G}_{ijk\ell}\pi^{k\ell} - \frac{\sqrt{g}}{16\pi G_N}\left(R - 2\Lambda\right), \tag{4.36}$$

where G_N is the Newton constant, and λ has been set equal to 1. The quantum version of this constraint yields the Wheeler-De Witt equation.

General relativity is fundamentally built on the principle of spacetime diffeomorphism invariance. One might therefore expect that the fist-class constraints $\mathcal{H}_i(t, \mathbf{x})$ and $\mathcal{H}_\perp(t, \mathbf{x})$ just confirm the naive expectation, and form the algebra of spacetime diffeomorphisms. However, as is well-known, it is not so: Under the Poisson bracket, the constraints of general relativity do not even close to form a Lie algebra. Their commutation relations are

$$\left[\int d^D\mathbf{x}\,\zeta(\mathbf{x})\mathcal{H}_\perp(\mathbf{x}), \int d^D\mathbf{y}\,\eta(\mathbf{y})\mathcal{H}_\perp(\mathbf{y})\right] = \int d^D\mathbf{x}\,(\zeta\partial_i\eta - \eta\partial_i\zeta)\,g^{ij}\mathcal{H}_j(\mathbf{x}), \tag{4.37}$$

$$\left[\int d^D\mathbf{x}\,\zeta^i(\mathbf{x})\mathcal{H}_i(\mathbf{x}), \int d^D\mathbf{y}\,\eta(\mathbf{y})\mathcal{H}_\perp(\mathbf{y})\right] = \int d^D\mathbf{x}\,\zeta^i\partial_i\eta\,\mathcal{H}_\perp(\mathbf{x}) \tag{4.38}$$

$$\left[\int d^D\mathbf{x}\,\zeta^i(\mathbf{x})\mathcal{H}_i(\mathbf{x}), \int d^D\mathbf{y}\,\eta^j(\mathbf{y})\mathcal{H}_j(\mathbf{y})\right] = \int d^D\mathbf{x}\,(\zeta^i\partial_i\eta^k - \eta^i\partial_i\zeta^k)\,\mathcal{H}_k(\mathbf{x}). \tag{4.39}$$

First, (4.39) is easy to interpret: It shows that the $\mathcal{H}_i$ constraints form the Lie algebra of generators of spatial diffeomorphisms, preserving the time foliation of the canonical formalism. Similarly, (4.38) simply indicates that $\mathcal{H}_\perp(\mathbf{y})$ transforms as a density under the spatial diffeomorphisms generated by $\mathcal{H}_i$. The subtlety occurs in the commutation relation of two $\mathcal{H}_\perp$: Because of the explicit presence of g^{ij} in (4.37), the structure "constants" are field-dependent, and strictly speaking, the constraints do not form a Lie algebra. This fact contributes to the notorious conceptual as well as technical difficulties in the process of quantization of the relativistic theory (see, e.g., [31, 32]).

In our nonrelativistic gravity, the structure of constraints is slightly different than in general relativity. If the lapse field N is restricted to a function of time only, the constraint algebra is generated by the momentum constraints $\mathcal{H}_i(t, \mathbf{x})$, which take the general relativistic form (4.35), and the integral of $\mathcal{H}$:

$$\mathcal{H}_0 \equiv \int d^D\mathbf{x}\,\mathcal{H}(t, \mathbf{x}). \tag{4.40}$$

It is easy to show that these constraints form a closed algebra. The commutator of two $\mathcal{H}_i(\mathbf{x})$ generators coincides with (4.39). Our $\mathcal{H}(\mathbf{x})$ transforms as a density under generators

$\mathcal{H}_i$ of spatial diffeomorphisms and therefore satisfies (4.38), implying that $\mathcal{H}_i(\mathbf{x})$ commute with the zero mode $\mathcal{H}_0$, as can be seen by setting $\eta = 1$ in (4.38). (For this to work, it is important that $\mathcal{H}(\mathbf{x})$ transforms as a density, which eliminates possible terms $\sim \eta \partial_i \zeta^i$ in (4.38).) Finally, $\mathcal{H}_0$ of course commutes with itself.

The general theory of constrained systems [33] can be used to predict the number of physical degrees of freedom in our system. There are $2D$ first-class constraints per spacetime point: D components of $\mathcal{H}_i$ and D momenta conjugate to N_i. We also have $D(D+3)$ fields: $D(D+1)/2$ components of g_{ij} and their conjugate momenta, and D components of N_i and their momenta. The expected number of degrees of freedom per spacetime point is

$$\#(\mathrm{DoF}) = \frac{1}{2} \left(\#(\text{field components}) - 2 \times \#(\text{first-class constraints}) \right) =$$
$$= \frac{D(D-1)}{2} = \frac{(D+1)(D-2)}{2} + 1. \tag{4.41}$$

The number of massless graviton polarizations in relativistic gravity in $D+1$ spacetime dimensions is $(D+1)(D-2)/2$. Thus, compared to general relativity, our theory is generically expected to have one additional propagating scalar degree of freedom, at least in the absence of any additional gauge symmetry.

4.5 At the free-field fixed point with $z = 2$

In order to prepare for the study of the full interacting theory, it is useful to first understand the properties of its free-field fixed point limit. Free-field limits of anisotropic theories with nontrivial dynamical critical exponent z exhibit interesting properties, such as families of inequivalent fixed points, as we have seen in the example of $z = 2$ Yang-Mills theory in [2].

4.5.1 Scaling properties and the critical dimension

By design, our nonrelativistic gravity has a free-field limit with anisotropic scaling of space and time, characterized by dynamical critical exponent $z = 2$. The engineering dimensions (i.e., the scaling dimensions at the $z = 2$ free-field fixed point) of various quantities are as follows. First, just as in general relativity, the metric components g_{ij} are naturally dimensionless as a result of their geometric origin. The dimensions of the remaining fields are then determined to be

$$[g_{ij}] = 0, \qquad [N_i] = 1, \qquad [N] = 0. \tag{4.42}$$

In the formulation that uses the auxiliary field B^{ij}, we also have $[B^{ij}] = 2$.

The coupling constants appearing in (4.22) have dimensions

$$[\kappa] = \frac{2-D}{2}, \qquad [\kappa_W] = \frac{2-D}{2}, \qquad [\lambda] = 0. \tag{4.43}$$

As in the system of the Lifshitz scalar at $z = 2$, making the gravity theory anisotropic with dynamical exponent $z = 2$ has shifted the critical dimension of the free-field fixed point, from $1 + 1$ to $2 + 1$. This is the dimension where both κ and κ_W are dimensionless. Of

course, in the critical dimension $D = 2$, the Einstein tensor and consequently the potential term S_K in the action vanish identically. This simplification of $z = 2$ gravity in the critical case of $2+1$ dimensions is closely related to the simplification of relativistic gravity in $1+1$ dimensions, where the Einstein-Hilbert action is a topological invariant.

The free-field fixed point is defined by "turning off" all the coupling constants that measure interactions. Our theory has three couplings: κ_W, κ and λ. As it turns out, only one of them measures the strength of self-interactions of the gravitons, and turning it off makes the theory free. More precisely, turning off the interactions is equivalent to sending κ_W to zero while keeping λ and the ratio

$$\gamma = \frac{\kappa}{\kappa_W} \tag{4.44}$$

fixed. This leaves two dimensionless coupling constants γ and λ which survive in the noninteracting limit and measure the properties of the free-field fixed point. Thus, we obtain a two-parameter family of fixed points, all with $z = 2$. This is very analogous to the case of quantum critical Yang-Mills theory studied in [2], which exhibits a similar one-parameter family of free fixed points with $z = 2$.

4.5.2 The spectrum

We will now determine the spectrum of physical excitations, and their dispersion relations, in the family of free fixed point parametrized by γ and λ.

The action at the free-field fixed point can be found by expanding the theory around the flat background with $g_{ij} = \delta_{ij}$, $N = 1$ and $N_i = 0$. This background is indeed a classical solution of the theory, for any value of γ and λ. We expand the fields around this solution, writing

$$g_{ij} = \delta_{ij} + \kappa_W h_{ij}. \tag{4.45}$$

N_i are of order κ_W, and we rescale them accordingly. Finally, the corrections to $N = 1$ drop out in this approximation.

The Gaussian action of the linearized theory is then

$$\begin{aligned}
S = \frac{1}{2} \int dt\, d^D\mathbf{x} \Bigg\{ &\frac{1}{\gamma^2} \left[\left(\dot{h}_{ij} - \partial_i N_j - \partial_j N_i \right) \left(\dot{h}_{ij} - \partial_i N_j - \partial_j N_i \right) - \lambda \left(\dot{h}_{ii} - 2\partial_i N_i \right)^2 \right] \\
&- \frac{\gamma^2}{16} h_{ij} \left[\frac{(D-2)(2\lambda-1)}{D\lambda-1} \left(\partial_i \partial_j \partial_k \partial_\ell + \delta_{ij} \delta_{k\ell} \left(\partial^2 \right)^2 - 2\delta_{ij} \partial_k \partial_\ell \partial^2 \right) \right. \\
&\left. + 2 \left(\delta_{ij} \partial_k \partial_\ell - \delta_{ik} \partial_j \partial_\ell \right) \partial^2 + \left(\delta_{ik} \delta_{j\ell} - \delta_{ij} \delta_{k\ell} \right) \left(\partial^2 \right)^2 \right] h_{k\ell} \Bigg\}.
\end{aligned} \tag{4.46}$$

In order to identify the propagating modes and determine their dispersion relations, we must make a suitable gauge choice and diagonalize this action. Given the nonrelativistic character of the theory, it is natural to choose

$$N_i = 0 \tag{4.47}$$

as our gauge-fixing condition. This gauge choice does not fix the gauge symmetries completely, leaving the group of time-independent spatial diffeomorphisms unfixed. In addition,

making this gauge choice implies that the fields in (4.46) are constrained by the following analog of the Gauss constraint,

$$\partial_i \dot{h}_{ij} = \lambda \partial_j \dot{h}, \tag{4.48}$$

where $h \equiv h_{ii}$. This constraint comes from the linearized equation of motion of N_i in the full gauge-invariant action. We can fix the residual gauge symmetry by setting

$$\partial_i h_{ij} - \lambda \partial_j h = 0, \tag{4.49}$$

at some fixed time surface, $t = t_0$. The constraint (4.48) then implies that (4.49) continues to hold at all times.

(4.49) is a legitimate gauge choice for values of λ not equal to one. When $\lambda = 1$, (4.49) is not attainable by a spatial diffeomorphism. The simplest way to see that is to apply ∂_j to (4.49). The left hand side then equals $\partial_j \partial_i h_{ij} - \partial^2 h = R$, the linearized Ricci scalar which cannot be set to zero by a gauge transformation. As we will see below, $\lambda = 1$ is indeed a special case, where the free-field fixed point exhibits an enhanced gauge symmetry.

In order to read off the physical polarizations of the metric and their dispersion relations, we need to rewrite the quadratic action (4.46) in variables that automatically take into account the constraints (4.48). We switch from h_{ij} to the new variables, defined as

$$H_{ij} = h_{ij} - \lambda \delta_{ij} h. \tag{4.50}$$

Our residual gauge fixing (4.49) together with the Gauss constraint (4.48) imply that H_{ij} is transverse,

$$\partial_i H_{ij} = 0. \tag{4.51}$$

The transverse tensor H_{ij} contains all the physical polarizations of the metric. In order to separate the individual modes, we further decompose H_{ij} into the transverse traceless part $\widetilde{H}_{ij}$ and the trace part H:

$$H_{ij} = \widetilde{H}_{ij} + \frac{1}{D-1} \left(\delta_{ij} - \frac{\partial_i \partial_j}{\partial^2} \right) H. \tag{4.52}$$

We have $\widetilde{H}_{ii} = 0$, $\partial_i \widetilde{H}_{ij} = 0$, and $H = H_{ii}$.

In this gauge, the linear equations of motion that follow from (4.46) can be diagonalized, and one can determine the number of physical polarizations and their dispersion relations. $\widetilde{H}_{ij}$ yields $(D-2)(D+1)/2$ transverse traceless polarizations, all with the same dispersion relation

$$\omega^2 = \frac{\gamma^4}{16} (\mathbf{k}^2)^2. \tag{4.53}$$

In addition, the trace H leads to one mode, whose dispersion relation is

$$\omega^2 = \frac{\Gamma^4}{16} (\mathbf{k}^2)^2, \tag{4.54}$$

with

$$\Gamma^4 = \frac{(D-2)^2 (\lambda-1)^2}{(D\lambda-1)^2} \gamma^4. \tag{4.55}$$

The free-field fixed point is well-defined in the large range of the parameters λ and D for which the energy of the excitations is bounded from below. We can identify this physical range of parameters by expressing the gauge-fixed action in terms of $\widetilde{H}_{ij}$ and H. The kinetic term becomes

$$S_K = \frac{1}{2\gamma^2} \int dt \, d^D\mathbf{x} \left\{ \dot{\widetilde{H}}_{ij} \dot{\widetilde{H}}_{ij} + \frac{\lambda - 1}{(D-1)(D\lambda - 1)} \dot{H}^2 \right\}, \tag{4.56}$$

and the potential term is

$$S_V = \frac{1}{32\gamma^2} \int dt \, d^D\mathbf{x} \left\{ \partial^2 \widetilde{H}_{ij} \partial^2 \widetilde{H}_{ij} + \frac{(D-2)^2(\lambda - 1)^3}{(D-1)(D\lambda - 1)^3} \left(\partial^2 H \right)^2 \right\}. \tag{4.57}$$

Hence, assuming $D > 1$ the energy of the physical modes is positive definite when $\lambda < 1/D$ or $\lambda > 1$. In the complementary regime $1/D < \lambda < 1$, the scalar mode H is a ghost.

The dispersion relation (4.54) for the scalar mode H suggests that something special happens at $\lambda = 1/D$ and $\lambda = 1$. When $\lambda = 1/D$, the De Witt metric develops a null direction. As a result, this is the value at which the theory may develop a local version of conformal symmetry, depending on the specific form of the potential term in the action. This case will be relevant to our membrane theory in section 5, where we will be interested in incorporating a local Weyl invariance in $2 + 1$ dimensions.

At the other special value, $\lambda = 1$, the equation of motion for the scalar mode $H(t, \mathbf{x})$ collapses to $\ddot{H} = 0$, with the general solution

$$H(t, \mathbf{x}) = H_0(\mathbf{x}) + t H_1(\mathbf{x}). \tag{4.58}$$

If present, such degrees of freedom would be difficult to interpret as physical excitations. As it turns out, at this value of λ, the linearized theory develops an enhanced gauge symmetry, acting via

$$\delta N_i = \partial_i \varepsilon(\mathbf{x}), \qquad \delta h_{ij} = 0, \tag{4.59}$$

i.e., as a time-independent U(1) gauge transformation. This is the Abelian symmetry (4.17), linearized and reduced to preserve $A = 0$. This spatial gauge symmetry plays an interesting role in the theory. The $N_i = 0$ gauge can now be attained in two steps, first by using a diffeomorphism to get $N_i = \partial_i u$ for some function $u(\mathbf{x})$, and then using (4.59) with $\varepsilon = -u$ to set $N_i = 0$. The first step leaves an extra unfixed diffeomorphism symmetry, given by $\zeta^i(t, \mathbf{x})$ that satisfy $\dot{\zeta}^i = \partial_i u(\mathbf{x})$. The generators of such unfixed diffeomorphisms are of the form

$$\zeta^i(t, \mathbf{x}) = \zeta_0^i(\mathbf{x}) + t \partial_i u(\mathbf{x}). \tag{4.60}$$

These residual diffeomorphisms acts on H via $\delta H \sim \partial_i \zeta^i$. The extra gauge freedom given by $u(\mathbf{x})$ can be used to set H_1 in (4.58) to zero, leaving the transverse traceless gravitons as the only physical excitations at $\lambda = 1$.

4.6 Relevant deformations: lower-dimension operators in the potential term

So far, we concentrated on the terms in the action which have the same engineering dimension as the kinetic term (4.1). These are the terms that determine the behavior of the $z = 2$

fixed point. Now we extend our analysis to incorporate operators with lower dimensions, compatible with the symmetries of the theory. If such operators exist, general arguments from effective field theory indicate that such terms will be generated by quantum effects, and will dominate over the original terms in S_V in the long-distance dynamics of the theory.

We will discuss the issue of relevant deformations of $z = 2$ gravity in $D + 1$ dimensions only briefly, because in section 5 we will follow a different route: We will impose an additional gauge symmetry, related to Weyl invariance, which will forbid any lower-dimensional operators in $z = 2$ gravity in $2 + 1$ dimensions.

In theories satisfying the detailed balance condition, there is a hierarchy of ways in which lower-dimension operators can be added to the classical theory:

1. In the minimal modification, we add lower-dimensional operators to W, and thus preserve the detailed balance condition.

2. We can add terms to E^{ij} which respect all the symmetries but cannot be derived from varying any action in D dimensions (if such terms exist).[6]

3. Finally, one can simply add lower-dimension operators directly to the action in $D + 1$ dimensions, softly breaking not only the condition of detailed balance, but also the fact that in the representation with the auxiliary field B, only terms at least linear in B appear in the action.

In the following, we will mostly focus on the first option, in which lower-dimensional terms are added to W. For $z = 2$ gravity without matter, the only such term that can be added to W is the cosmological constant term. Restoring the cosmological constant in (4.9),

$$W = \frac{1}{\kappa_W^2} \int d^D\mathbf{x}\, \sqrt{g}(R - 2\Lambda_W),$$
(4.61)

we get the following action in $D + 1$ dimensions

$$S = \frac{1}{2} \int dt\, d^D\mathbf{x}\, \sqrt{g} \left\{ \frac{1}{\kappa^2 N} \left(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i \right) G^{ijk\ell} \left(\dot{g}_{k\ell} - \nabla_k N_\ell - \nabla_\ell N_k \right) \right.$$
$$\left. - \frac{\kappa^2 N}{4\kappa_W^4} \left(R^{ij} - \frac{1}{2}Rg^{ij} + \Lambda_W g^{ij} \right) \mathcal{G}_{ijk\ell} \left(R^{k\ell} - \frac{1}{2}Rg^{k\ell} + \Lambda_W g^{k\ell} \right) \right\}.$$
(4.62)

In dimensions $D > 2$, turning on Λ_W in W induces two new terms in S_V: The spatial Ricci scalar term R and the spatial volume term. In $2 + 1$ dimensions, since $R^{ij} - Rg^{ij}/2$ vanishes identically, no Ricci scalar term is produced in S_V. We will return to this case in detail in section 5, and limit our present discussion to $D > 2$.

It is natural to define a scale M,

$$M^2 = \frac{D - 2}{1 - D\lambda} \Lambda_W.$$
(4.63)

In terms of M, the action becomes

$$S = \frac{1}{2} \int dt\, d^D\mathbf{x}\, \sqrt{g} \left\{ \frac{1}{\kappa^2 N} \left(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i \right) G^{ijk\ell} \left(\dot{g}_{k\ell} - \nabla_k N_\ell - \nabla_\ell N_k \right) \right.$$
$$\left. - \frac{\kappa^2 N}{4\kappa_W^4} \left(R^{ij} R_{ij} + a R^2 - M^2 R + \frac{D(1 - D\lambda)}{(D-2)^2} M^4 \right) \right\}. \tag{4.64}$$

The constant a takes the value given in (4.11).

For simplicity, we will assume $M^2 > 0$. If $\lambda > 1/D$, this means starting with a negative cosmological constant, $\Lambda_W < 0$, in W. The other sign of M^2 would correspond to the gravity analog of the "spatially modulated" phases in the Lifshitz scalar theory, which we will not study in this paper.

Under the influence of the deformation by lower-dimension operators, the theory will flow from $z = 2$ at short distances, to $z = 1$ in the infrared. This flow to $z = 1$ is in fact generic for quantum theories of the Lifshitz type (see [2]). The dynamics of the theory at long distances will be dominated by the most relevant operators. In our gravity theory, those will be the terms in S_V with couplings involving nonzero powers of M: The spatial Ricci scalar, and the spatial volume term. Together with the kinetic term S_K, these are exactly the ingredients that are required in general relativity.

In order to compare the long-distance physics of the theory deformed by relevant operators to that of Einstein's theory, it is natural to redefine the time coordinate,

$$x^0 = ct, \qquad c = \frac{\gamma^2}{4} M. \tag{4.65}$$

This is one of the most notable features of our construction: The effective long-distance speed of light originates microscopically from a relevant coupling in the theory describing the anisotropic short-distance dynamics.

In these relativistic coordinates, the dominant long-distance terms in the Hamiltonian $\mathcal{H}(\mathbf{x})$ of our deformed theory are precisely such that they reproduce the relativistic Hamiltonian $\mathcal{H}_\perp(\mathbf{x})$ of (4.36), with the effective Newton constant

$$G_N = \frac{\kappa_W^2}{8\pi M}, \tag{4.66}$$

and the effective cosmological constant

$$2\Lambda = \frac{D(1 - D\lambda)}{(D-2)^2} M^2 = \frac{D}{D-2} \Lambda_W. \tag{4.67}$$

Thus, we conclude that

- under the influence of relevant deformations, the anisotropic gravity theory flows in the infrared limit naturally to a theory with isotropic scaling and $z = 1$, and leads to long-distance physics which is remarkably close to general relativity.

- There are several differences between the general relativity and the $z = 1$ infrared limit of our theory. First, our Hamiltonian depends on the additional coupling λ, which equals 1 in general relativity. In addition, we restricted the lapse variable N to be independent of spatial coordinates.

- Notably, the emerging long-distance speed of light (4.65), the effective Newton constant (4.66), and also the effective cosmological constant (4.67) all originate from the relevant deformations of a deeply nonrelativistic short-distance theory of gravity with anisotropic scaling and $z = 2$.

- While interactions and quantum effects will affect some features of the flow, our conclusions are exact in the noninteracting limit $\kappa_W = 0$.

5 Membranes at criticality: $z = 2$ gravity and matter in 2+1 dimensions

Having presented the construction of $z = 2$ gravity, we can now return to our original problem, and combine this theory in $2 + 1$ dimensions with Lifshitz matter, in order to establish the desired connection to the partition function of the bosonic string. In the process, we must clarify how the worldsheet Weyl invariance of critical string theory can be incorporated into our scheme.

5.1 Coupling $z = 2$ gravity to Lifshitz matter

We now consider the $z = 2$ gravity theory in its critical dimension $2 + 1$, coupled to 26 Lifshitz scalar fields $X^I(t, \mathbf{x})$, $I = 1, \ldots 26$. Our starting point is the Polyakov worldsheet action for the bosonic string of Euclidean worldsheet signature, embedded in the spacetime target manifold $\mathbf{R}^{26}$ parametrized by coordinates X^I and equipped with the flat Euclidean metric δ_{IJ}:

$$W = \frac{1}{4\pi\alpha'} \int d^2\mathbf{x} \sqrt{g}\, g^{ij} \partial_i X^I \partial_j X^I. \tag{5.1}$$

Combining the construction of $z = 2$ gravity presented in section 4 with the Lifshitz matter reviewed in section 3.1, we obtain the action of the coupled system of $z = 2$ gravity and $z = 2$ matter in $2 + 1$ dimensions,

$$S = \frac{1}{2} \int dt\, d^2\mathbf{x} \sqrt{g} \left\{ \frac{1}{\kappa^2 N} \left(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i \right) G^{ijk\ell} \left(\dot{g}_{k\ell} - \nabla_k N_\ell - \nabla_\ell N_k \right) \right.$$
$$\left. + \frac{1}{\alpha_M N} \left(\dot{X} - N^i \partial_i X \right)^2 - \frac{\alpha_M N}{(4\pi\alpha')^2} (\Delta X)^2 - \frac{\kappa^2 N}{4(4\pi\alpha')^2} T^{ij} \mathcal{G}_{ijk\ell} T^{k\ell} \right\}. \tag{5.2}$$

Here

$$\Delta X^I \equiv \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} \partial_j X^I) \tag{5.3}$$

is the Laplace operator of g_{ij} acting on the scalar field X^I, and

$$T_{ij} \equiv 4\pi\alpha' \frac{1}{\sqrt{g}} \frac{\delta W}{\delta g^{ij}} = \partial_i X^I \partial_j X^I - \frac{1}{2} g_{ij} (g^{k\ell} \partial_k X^I \partial_\ell X^I) \tag{5.4}$$

is the energy-momentum tensor of the scalar fields in (5.1). This action is gauge invariant under foliation-preserving diffeomorphisms, and satisfies the detailed balance condition with respect to (5.1). Under the foliation-preserving diffeomorphisms (4.13), the Lifshitz scalars transform as

$$\delta X^I = f \dot{X}^I + \zeta^i \partial_i X^I. \tag{5.5}$$

Note that the kinetic term for the Lifshitz scalars in (5.2) required the introduction of a new coupling α_M of the same dimension as α', i.e., spacetime length squared. We can express this new spacetime length scale in terms of α', and define a new dimensionless parameter

$$\kappa_M^2 = \frac{\alpha_M}{4\pi\alpha'} \tag{5.6}$$

instead. In addition to κ_M, the theory has two other dimensionless couplings: λ, which is hidden in the definition of the De Witt metric (4.2), and κ. Since the coupling of matter to gravity leads to the nonlinear terms $T_{ij}T^{ij}$ in (5.2), the theory is interacting at nonzero κ. The free limit corresponds to taking $\alpha' \to 0$ and $\kappa \to 0$. The remaining two dimensionless parameters κ_M and λ survive in the free field limit, and characterize the properties of the family of free-field fixed points in $z = 2$ gravity with matter in $2 + 1$ dimensions.

5.2 Anisotropic Weyl symmetry

The gauge symmetries of our coupled theory (5.2) do not yet match those of critical string theory. Upon canonical quantization, worldsheet diffeomorphisms are reproduced as symmetries of wavefunctions, but Weyl invariance is not.

One could go in the direction of noncritical string theory, and try to develop a corresponding noncritical theory of membranes.[7] In this paper, we are more interested in reproducing the conventional critical bosonic string, and we must therefore look for an implementation of a $2 + 1$ dimensional analog of Weyl invariance on the membrane worldvolume, as an additional gauge symmetry supplementing the foliated diffeomorphisms.

The requirement of a local Weyl invariance will actually fix the value of λ of the gravity sector uniquely. Moreover, this gauge invariance extends to the matter sector as well, described by the Lifshitz scalar theory. We define the "anisotropic Weyl transformations" — for any value of the dynamical critical exponent z — as follows,

$$g_{ij} \to \exp\{2\Omega(t, \mathbf{x})\}\, g_{ij}, \qquad N_i \to \exp\{2\Omega(t, \mathbf{x})\}\, N_i, \qquad N \to \exp\{z\Omega(t, \mathbf{x})\}\, N. \tag{5.7}$$

Since the anisotropic Weyl transformations act nontrivially on N, we can no longer restrict N to be independent of space; N is now a $2 + 1$ dimensional field, $N(t, \mathbf{x})$.

Such anisotropic Weyl transformations with fixed z form a closed algebra with foliation-preserving diffeomorphisms (4.16): Denoting by δ_ω the infinitesimal Weyl transformation with parameter $\omega(t, \mathbf{x})$, and by δ_v the infinitesimal foliation-preserving diffeomorphism transformation $v \equiv (f(t), \zeta^i(t, \mathbf{x}))$ as given in (4.16), one can show that their commutator yields another anisotropic Weyl transformation,

$$[\delta_v, \delta_\omega] = \delta_{\widetilde{\omega}}, \qquad \text{with } \widetilde{\omega} = \zeta^i \partial_i \omega + f\, \dot{\omega}. \tag{5.8}$$

Specializing to $z = 2$, our Lagrangian is classically invariant under the anisotropic Weyl transformations if we set $\lambda = 1/2$. In the proof of this gauge invariance, it is important

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[7]This could be relevant for the relation between noncritical strings in two dimensions, noncritical M-theory in $2 + 1$ dimensions, and topological strings of the A-model, as discussed in [8]. This possibility was indeed one of the original motivations for this project.

that for infinitesimal Weyl transformation ω (and again temporarily restoring arbitrary z and arbitrary space dimension D for future reference)

$$\delta_\omega(\nabla_i N_j) = 2\omega\nabla_i N_j + N_j\partial_i\omega - N_i\partial_j\omega + g_{ij}g^{k\ell}N_k\partial_\ell\omega. \tag{5.9}$$

Contracting this with the De Witt metric, we get

$$G^{ijk\ell}\delta_\omega(\nabla_i N_j) = 2\omega\, G^{ijk\ell}\nabla_i N_j + (1 - D\lambda)g^{k\ell}g^{ij}N_i\partial_j\omega. \tag{5.10}$$

Hence, for the conformal value $\lambda = 1/D$, the terms with derivatives of $\delta\omega$ vanish, and the kinetic term for gravity will be invariant under the local anisotropic Weyl transformations.

Returning now to the case of interest, $D = 2$, we set the coupling constant λ in the De Witt metric $G^{ijk\ell}$ equal to its conformal value $\lambda = 1/2$. Our action (5.2) is gauge invariant under local anisotropic Weyl transformations, at least at the classical level. This continues to be the case after coupling to the Lifshitz scalars X^I, provided they transform with weight zero under the Weyl transformations, $\delta_\omega X^I = 0$. The requirement of local anisotropic Weyl symmetry forbids any relevant terms in the action of our coupled system of $z = 2$ gravity and $z = 2$ matter in $2 + 1$ dimensions.

5.3 Canonical formulation

In order to understand properties of the ground-state wavefunction, we would like to quantize our $2 + 1$ dimensonal theory with anisotropic Weyl invariance canonically on $\Sigma_h \times \mathbf{R}$, where Σ_h is the Riemann surface of genus h.

We use the ADM formulation of section 4.4, generalized to the presence of matter. The momenta conjugate to g_{ij} were found in (4.30):

$$\pi^{ij} = \frac{\sqrt{g}}{\kappa^2 N}G^{ijk\ell}\left(\dot{g}_{k\ell} - \nabla_k N_\ell - \nabla_\ell N_k\right) = \frac{2\sqrt{g}}{\kappa^2}G^{ijk\ell}K_{k\ell}. \tag{5.11}$$

Once we set $\lambda = 1/2$ in the De Witt metric, as required by anisotropic Weyl invariance, we find that the momenta (5.11)

$$\pi^{ij} = \frac{2\sqrt{g}}{\kappa^2}\left(K^{ij} - \frac{1}{2}g^{ij}K\right), \tag{5.12}$$

(where $K \equiv g^{ij}K_{ij}$) are traceless,

$$\pi_i^i \equiv g_{ij}\pi^{ij} = 0, \tag{5.13}$$

as a consequence of the local Weyl symmetry. (5.13) is a new primary constraint, When this constraint is solved, only the traceless momenta — which we denote by $\widetilde{\pi}^{ij}$ — appear in the theory.

Similarly, the momenta P_I conjugate to the Lifshitz scalars X^I are

$$P_I = \frac{\delta S}{\delta\dot{X}^I} = \frac{\sqrt{g}}{\alpha_M N}(\dot{X}^I - N^i\partial_i X^I). \tag{5.14}$$

The Hamiltonian is

$$H = \int d^2\mathbf{x} \left(N\mathcal{H} + N_i\mathcal{H}^i \right), \tag{5.15}$$

with

$$\mathcal{H} = \frac{\kappa^2}{2\sqrt{g}}\widetilde{\pi}^{ij}g_{ik}g_{j\ell}\widetilde{\pi}^{k\ell} + \frac{\alpha_M}{2\sqrt{g}}P_I P_I + \frac{\sqrt{g}}{2(4\pi\alpha')^2}\left(\alpha_M \Delta X^I \Delta X^I + \frac{\kappa^2}{4}T_{ij}T^{ij} \right), \tag{5.16}$$

and

$$\mathcal{H}^i = -2\nabla_j\pi^{ij} + g^{ij}P_I\,\partial_j X^I. \tag{5.17}$$

5.4 The algebra of constraints

The anisotropic Weyl invariance requires N to be a general function of t and $\mathbf{x}$. As a result, the structure of the Hamiltonian (5.15) indicates that in the Weyl invariant theory, both $\mathcal{H}_i(\mathbf{x})$ and $\mathcal{H}(\mathbf{x})$ (and not just its zero mode $\mathcal{H}_0 = \int d^2\mathbf{x}\,\mathcal{H}(\mathbf{x})$) will play the role of the constraints, and we must determine their algebra.

As an alternative to the general relativistic constraints (4.37)–(4.39), another algebra of "general covariance" was proposed in [34]:

$$\left[\int d^D\mathbf{x}\,\zeta(\mathbf{x})\mathcal{H}_\perp(\mathbf{x}), \int d^D\mathbf{y}\,\eta(\mathbf{y})\mathcal{H}_\perp(\mathbf{y}) \right] = 0, \tag{5.18}$$

$$\left[\int d^D\mathbf{x}\,\zeta^i(\mathbf{x})\mathcal{H}_i(\mathbf{x}), \int d^D\mathbf{y}\,\eta(\mathbf{y})\mathcal{H}_\perp(\mathbf{y}) \right] = \int d^D\mathbf{x}\,\zeta^i\partial_i\eta\,\mathcal{H}_\perp(\mathbf{x}), \tag{5.19}$$

$$\left[\int d^D\mathbf{x}\,\zeta^i(\mathbf{x})\mathcal{H}_i(\mathbf{x}), \int d^D\mathbf{y}\,\eta^j(\mathbf{y})\mathcal{H}_j(\mathbf{y}) \right] = \int d^D\mathbf{x}\,(\zeta^i\partial_i\eta^k - \eta^i\partial_i\zeta^k)\,\mathcal{H}_k(\mathbf{x}). \tag{5.20}$$

This in some sense is a nicer symmetry than (4.37)–(4.39): It actually forms a Lie algebra, with structure constants independent of the fields. It is a symmetry of the so-called "ultralocal theory of gravity" [34, 35] which in fact fits naturally into our framework: The action in the ultralocal theory of gravity also takes the form $S = S_K - S_V$, with S_K given by (4.1), and S_V set equal to zero (or to $\sqrt{g}\Lambda$ in the case of a nonzero cosmological constant).

As it turns out, the ultralocal algebra (5.18)–(5.20) is also the algebra of Hamiltonian constraints of $z = 2$ gravity in $2+1$ dimensions with Weyl invariance and without matter. The simplest way to see that is to notice that in the critical dimension $D = 2$, the potential term S_V in our $z = 2$ theory vanishes identically, and the full action coincides that that of the ultralocal theory, with $\lambda = 1/2$.

When Lifshitz matter is introduced, the commutator of two $\mathcal{H}(\mathbf{x})$ no longer vanishes. Instead, we get

$$\left[\int d^2\mathbf{x}\,\zeta(\mathbf{x})\mathcal{H}(\mathbf{x}), \int d^2\mathbf{y}\,\eta(\mathbf{y})\mathcal{H}(\mathbf{y}) \right] = \int d^2\mathbf{x}\,(\zeta\partial_i\eta - \eta\partial_i\zeta)\,\Phi^i(\mathbf{x}), \tag{5.21}$$

with

$$\Phi^i(\mathbf{x}) = -\frac{\alpha_M}{(4\pi\alpha')^2}\left(\kappa^2\widetilde{\pi}^{ij}\partial_j X^I \Delta X^I - \frac{\kappa^2}{2}P^I\partial_j X^I T^{ij} + \alpha_M\,g^{ij}(P^I\partial_j\Delta X^I - \partial_j P^I\Delta X^I) \right). \tag{5.22}$$

One could attempt to add $\Phi^i(\mathbf{x})$ to the list of constraints, and continue the process until the constraint algebra closes. However, there is a simpler alternative, which will go a long way towards solving our original problem. Note first that Φ^i can be rewritten as

$$\Phi^i(\mathbf{x}) = \frac{i\alpha_M}{(4\pi\alpha')^2} \left\{ \kappa^2 \Delta X^I \partial_j X^I \left(i\widetilde{\pi}^{ij} - \frac{\sqrt{g}}{8\pi\alpha'} T^{ij} \right) \right.$$
$$\left. + \left(\alpha_M g^{ij} (\partial_j \Delta X^I - \Delta X^I \partial_j) - \frac{\kappa^2}{2} T^{ij} \partial_j X^I \right) \left(iP^I - \frac{\sqrt{g}}{4\pi\alpha'} \Delta X^I \right) \right\} . \quad (5.23)$$

This suggests introducing

$$a^{ij} = i\pi^{ij} + \frac{1}{2} \frac{\delta W}{\delta g_{ij}} = i\pi^{ij} - \frac{\sqrt{g}}{8\pi\alpha'} T^{ij}, \quad (5.24)$$

$$Q^I = iP^I + \frac{1}{2} \frac{\delta W}{\delta X^I} = iP^I - \frac{\sqrt{g}}{4\pi\alpha'} \Delta X^I, \quad (5.25)$$

and their complex conjugates

$$\overline{a}^{ij} = -i\pi^{ij} - \frac{\sqrt{g}}{8\pi\alpha'} T^{ij}, \qquad \overline{Q}^I = -iP^I - \frac{\sqrt{g}}{4\pi\alpha'} \Delta X^I. \quad (5.26)$$

In our system of gravity coupled to matter, Q^I and a^{ij} are the precise analogs of the $\mathcal{Q}$ variable (3.9) defined in our discussion of the Lifshitz scalar theory in section 3.1. In terms of these variables, the Hamiltonian constraint itself can be written as

$$\mathcal{H} = \frac{\kappa^2}{2\sqrt{g}} \overline{a}^{ij} G_{ijk\ell} a^{k\ell} + \frac{\alpha_M}{2\sqrt{g}} \overline{Q}^I Q^I. \quad (5.27)$$

Given these facts, the following way towards quantization of the system suggests itself. Instead of $\{\mathcal{H}_i, \mathcal{H}, \Phi^i, \ldots\}$, we can choose the constraints to be $\{\mathcal{H}_i, a^{ij}, Q^I\}$. This may not be the unique possibility how to approach the quantization of our system, but it does exhibit the following attractive features:

- Since $\mathcal{H}(\mathbf{x})$, $\mathcal{H}_i(\mathbf{x})$ and Φ^i are linear in a^{ij} and Q^I, the vanishing of our constraints a^{ij} and Q^I implies the vanishing of the Hamiltonian and momentum constraints $\mathcal{H}$ and $\mathcal{H}_i$, as well as Φ^i. Similarly, it implies that the constraint of (5.13) also vanishes, because $g_{ij} a^{ij} = i\pi^i_i$.

- a^{ij}, Q^I and $\mathcal{H}_i$ form a closed algebra of first-class constraints. First, a^{ij} and Q^I all commute. Moreover, their commutator with $\mathcal{H}_i$ simply states how a^{ij} and Q^I transform under spatial diffeomorphisms, and therefore vanishes when the constraints are satisfied.

Quantum mechanically, the physical wavefunctions of the membrane states should be annihilated by all the constraints. Our intended ground-state wavefunction

$$\Psi_0[g_{ij}(\mathbf{x}), X^I(\mathbf{x})] = \exp \left\{ -\frac{1}{8\pi\alpha'} \int d^2\mathbf{x} \, \sqrt{g} g^{ij} \partial_i X^I \partial_j X^I \right\} \quad (5.28)$$

satisfies the quantum version of the constraint equations,

$$a^{ij}\Psi_0 \equiv \left(\frac{\delta}{\delta g_{ij}} - \frac{\sqrt{g}}{8\pi\alpha'}T^{ij}\right)\Psi_0 = 0,$$
$$Q^I\Psi_0 \equiv \left(\frac{\delta}{\delta X^I} - \frac{\sqrt{g}}{4\pi\alpha'}\Delta X^I\right)\Psi_0 = 0, \tag{5.29}$$

as well as $\mathcal{H}_i\Psi_0 = 0$. It appears to be the only normalizable wavefunction satisfying all the constraints. As a result, the spectrum of membrane states will contain only the ground state, and no physical excited states.

This indicates that the quantization with this strong set of constraints provides an affirmative answer to the original question, about the existence of a membrane theory whose ground-state wavefunction on a Riemann surface Σ_h reproduces the partition function of the bosonic string on Σ_h.

5.5 Generalizations

The set of constraints which we imposed in the previous section is almost certainly unnecessarily strong. However, it does lead to the desired result, a membrane theory which reproduces the string partition functions. The resulting membrane theory therefore represents a solution of the first-quantized version of the auxiliary problem posed in the introduction, albeit perhaps not the most exciting one: The only physical excitation of the membranes are their ground states.

Here we present a few preliminary remarks which might be useful in trying to find more interesting realizations, with physical membrane states beyond the ground state.

In the theory without Weyl invariance, studied in section 4, it was natural to treat N as a function of only t, which resulted in a simple algebra of constraints. Since Weyl transformations act on N, in a theory with Weyl symmetry N must be allowed to depend on t and $\mathbf{x}$. However,

$$\widetilde{N} = \frac{N}{\sqrt{g}} \tag{5.30}$$

is an invariant under the Weyl transformations, and we may attempt to restrict $\widetilde{N}$ to be a function of only t. Such a restriction would not be fully invariant under all diffeomorphisms, but only under those that satisfy

$$\partial_i\zeta^i(t,\mathbf{x}) = 0. \tag{5.31}$$

These are the area-preserving diffeomorphisms of Σ. Under this restriction, the algebra of Hamiltonian constraints again closes on $\mathcal{H}_i(t,\mathbf{x})$ and $\mathcal{H}_0$.

This scenario is closely related to the possibility of not imposing Weyl invariance. This in turn implies that we can move away from the conformal value of the coupling, $\lambda = 1/2$. As we saw in section 4.5.2, this set of gauge symmetries leads to one additional degree of freedom, the conformal factor ϕ of the metric. In conformal gauge, we can write $g_{ij} = e^\phi\delta_{ij}$. In string theory, ϕ is known as the Liouville field. The kinetic term for this extra Liouville scalar ϕ will be of second order in time derivatives, $\sim (\dot{\phi})^2$. The analysis of section 4.5.2 shows that as we move away from the conformal value $\lambda = 1/2$ to larger λ, the sign of the

kinetic term for ϕ is negative. Thus, in this range of λ, ϕ plays the role of an extra target dimension, a phenomenon reminiscent of the behavior of the Liouville mode in noncritical string theory. The vanishing of the Hamiltonian constraint $\mathcal{H}_0$ on physical states will then impose an on-shell condition, which can be solved by membrane states with non-zero frequencies and non-zero spatial momenta in the target spacetime with coordinates (X^I, ϕ). In addition, if the worldsheet theory described by the classical Polyakov action W has a conformal anomaly (as in the case of noncritical string theory), the effective action in two dimensions contains a nonlocal term

$$W' \sim \int d^2\mathbf{x}\, R\frac{1}{\Delta}R. \tag{5.32}$$

In conformal gauge, $\delta W'/\delta\phi \sim \Delta\phi$. As a result, when we apply the logic of our construction to $W + W'$, the anomalous term W' will give rise to a nonzero contribution $\sim (\Delta\phi)^2$ in the potential term S_V of the $2 + 1$ dimensional action. In conformal gauge, the Liouville conformal factor thus becomes a full-fledged Lifshitz scalar.

Note that the effective metric on the target manifold (X^I, ϕ) will be relativistic, as can be seen from the worldvolume kinetic terms for these fields, schematically of the form $\dot{X}^2 - \dot{\phi}^2$. If such a theory can be consistently quantized, it is likely to produce a relativistic spectrum of low-frequency modes, for which we would have a natural interpretation, as the superfuid excitations of the second-quantized Bose-Einstein condensate discussed in section 2.

The full quantum theory of membranes in the noncritical regime, with the Liouville field ϕ as one of the dynamical degrees of freedom and playing the role of time, is likely to be very difficult to analyze, with complications similar to those that occur in string theory away from its critical dimension.

6 Conclusions

In this paper, we have introduced a new class of nonrelativistic gravity models, characterized by anisotropic scaling between space and time with a nontrivial value of the dynamical critical exponent $z = 2$. This anisotropy leads to a change of the critical dimension of the system to $2 + 1$, and makes the theory suitable for the worldvolume of a membrane where it can be coupled to quantum critical matter with $z = 2$.

Any mathematically consistent theory of gravity can be expected to have at least four different categories of applications:

 (i) On worldvolumes of strings or branes, as required by their worldvolume reparametrization invariance.

 (ii) As a theory of the observed gravitational effects in our Universe.

 (iii) In the context of the AdS/CFT correspondence, as a candidate for the dual description of interesting classes of CFTs and more general quantum field theories.

 (iv) Applications in mathematics, such as those produced by topological gravity and topological strings.

The present paper mostly focused on the first class of applications of nonrelativistic gravity, as a candidate theory on the membrane worldvolume where the $z = 2$ system is at its critical dimension. However, our more general discussion of gravity at $z = 2$ in $D + 1$ spacetime dimensions in section 4 can be expected to be useful for possible applications (ii) and (iii) as well (see also [36]).

As to (iv), we have seen in section 4.3.4 that gravity at the $z = 2$ Lifshitz point is intimately related to the Ricci flow equations, and in a sense represents the natural quantum field theory associated with the Ricci flow.[8] The concept of the Ricci flow was instrumental in Perelman's theory and the proof of the Poincaré conjecture [30]. It would be interesting to develop this connection further, and see for example whether correlation functions of natural observables in our field theory shed additional light on Perelman's theory. Our theories of gravity with anisotropic scaling should also be relevant to the mathematically rich theory of foliations and their invariants [19–21].

In the context of $z = 2$ worldvolume gravity, the problem of summing over membrane topologies and organizing the sum into a topological expansion is also put in a new light: The 3-manifolds in question now carry an additional topological structure of a foliation. It is possible that this extra structure makes the summation over a specific class of foliated manifolds more managable that the sum over all topologies. When membranes interact, the topology of the spatial leaves of the foliation changes. Hence, in the sum over topologies, membrane interactions are likely to require foliations with singularities, such as those that occur in Morse theory [38], with the role of the Morse function played by worldvolume time.

In this paper, we only considered the simplest case, of the bosonic theory. In order to see whether the ideas of anisotropic worldvolume gravity are relevant to the relativistic M2-branes of supersymmetric M-theory, a generalization of our framework to membranes with spacetime supersymmetry would be required. In particular, it is natural to ask whether any version of anisotropic gravity in $2 + 1$ dimensions can flow naturally to $z = 1$ at long distances and serve as a UV completion of the relativistic worldvolume theory [39] on the membranes of M-theory.

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[8]Another relation between Perelman's theory and quantum field theory was explored by Tseytlin [37].

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